On taking correlations into account in least-squares adjustment with a diagonal covariance matrix

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Motivation and Introduction

• Due to computational burden or modelization uncertainty, correlations are not taken into account in least-squares adjustment, leading to:

✓ an overestimation of the precision
✓ possibly unreliable estimates when data are strongly correlated

• Based on the work of Luati and Proietti (2011), we propose a way to take correlations into account in least-squares adjustments thanks to an diagonal covariance matrices. Mathematical equivalence is given for the mean estimator; results being extended for other classes of design matrices such as those used in GPS adjustment.
• Our study is based on simulations with the Mátern covariance family which allows a great flexibility. Consequently, the correlation structure of the data is exactly known. This allows us to compare the least-squares results by changing the covariance matrices. Our new weighting shows promising adequation with the true value of the estimates that are given by the original covariance matrices.

Mathematical background

The functional model is given by **y**=**Ax**+**ε**, where **A** is the design matrix, **y** the vector of observations, **x** the parameter vector, $\mathbf{W}_0$ the positive definite fully populated variance-covariance matrix of the observations, σ^2 the apriori variance factor, and **ε** the observation error vector with

$$E(\boldsymbol{\varepsilon}) = \mathbf{0},\ E(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^T) = \sigma^2 \mathbf{W}_0$$

The Generalized Least-Squares Estimator (GLSE) reads:

$$\hat{\mathbf{x}} = \left(\mathbf{A}^T \mathbf{W}_0^{-1} \mathbf{A}\right)^{-1} \mathbf{A}^T \mathbf{W}_0^{-1} \mathbf{y}$$

$\mathbf{W}_0$ can be estimated based on the residuals but is often missspecified: the correlation structure is not accurately known and diagonal matrices are preferred.
The difference of the estimate by taking **W** instead of $\mathbf{W}_0$ can be quantified (Kutterer 1999):

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}_0 - \left(\mathbf{A}^T \mathbf{W}^{-1} \mathbf{A}\right)^{-1} \mathbf{A}^T \Delta\mathbf{P} \mathbf{v}_0 = \hat{\mathbf{x}}_0 + \Delta\mathbf{x},$$

$$\mathbf{P} = \mathbf{P}_0 + \Delta\mathbf{P},\ \mathbf{P}_0 = \mathbf{W}_0^{-1},\ \mathbf{P} = \mathbf{W}^{-1}$$

This formula is based on the observations vector and difficult to estimate concretely, i.e. bounds are often proposed.

Equivalent diagonal kernel - mean estimator

• Equivalent diagonal kernels allow to take correlations into account in regression least-squares analysis thanks to diagonal covariance matrices (Luatti and Proietti 2011).
• For the mean estimator case **A=1**:

A necessary and sufficient condition for the equivalence between Generalized Least-Squares Estimator (GLSE) and Diagonal Least-Squares Estimator (DLSE) is that each element of the diagonal matrix is the sum of the row elements of the inverse of the covariance matrix

Equivalence is mathematically given for the estimate and the cofactor matrix of the unknown but NOT for the aposteriori variance factor of the observations.

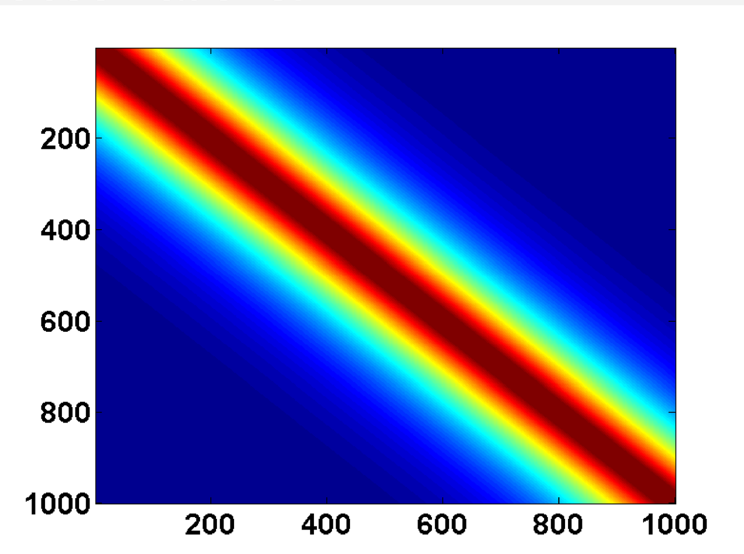

Fig.1 Covariance matrix of the observations (Mátern covariance function with smoothness 1 and inverse of the correlation length 0.01)

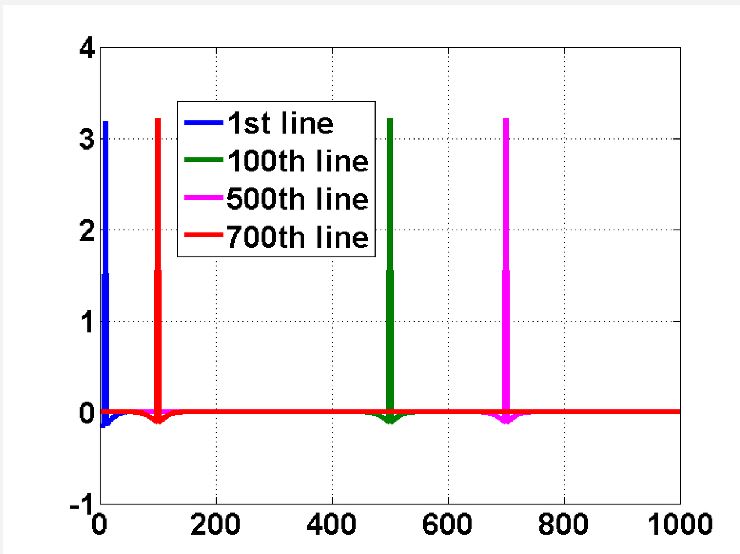

Fig.2 Four lines of the inverse of the covariance matrix

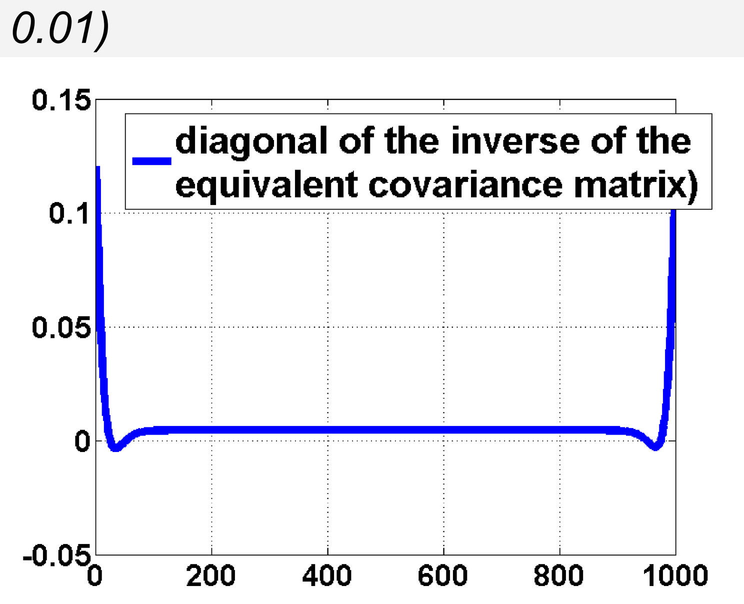

Fig.3 Diagonal of the inverse of the equivalent matrix

The sum of the elements of the line of the inverse gives the diagonal element of the equivalent kernel. The observations are having different weightings thanks to the equivalent kernel allowing correlations to be taken into account in the least-squares procedure.

3 kinds of covariance matrices were compared in a mean estimator case with simulated correlated time series (Mátern family):

	Estimate	Cofactor matrix of the estimate	A posteriori variance factor of the observations
W_0: fully populated covariance matrix used to simulate the time series (eigenvalue decomposition of W_0 with a N(0,1) random vector)	0.1728	0.1878	0.9999
W_equi: equivalent diagonal kernel	0.1728	0.1878	0.0834
W_diag: diagonal of W_0, in this case the identity matrix	0.0010 (i.e. the mean of the random vector)	0.2298	1.0508

Tab.1 Comparison of covariance matrices–mean estimator case with correlated data

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Equivalent diagonal kernel - point positioning

The results for the mean estimator case were extended to classes of design matrices such as those used in Point Positioning. For theses simulations, a real constellation was chosen with 4 satellites at different elevations. The correlation of the simulated time series were chosen to follow the model proposed by Kermarrec and Schön (2014). This model is based on the Mátern covariance family; the weighting being elevation dependent. An eigenvalue decomposition was used to simulate the corresponding time series.

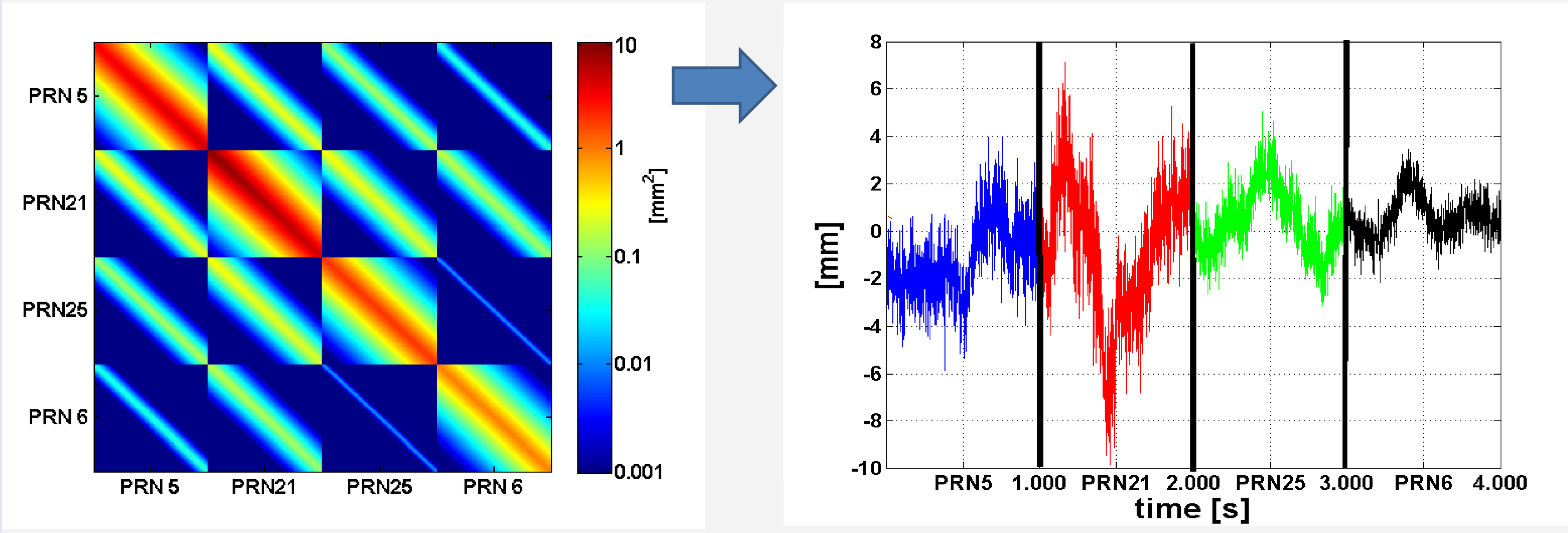

Fig.4 Covariance matrix with 4 satellites and the corresponding time series

The design matrix for the least-squares adjustment is showing a slight linear dependency with time (1000 epochs à 30s per satellite). Although no mathematical equivalence is given anymore, the results for the equivalent kernel are up to 20% better than the diagonal model.

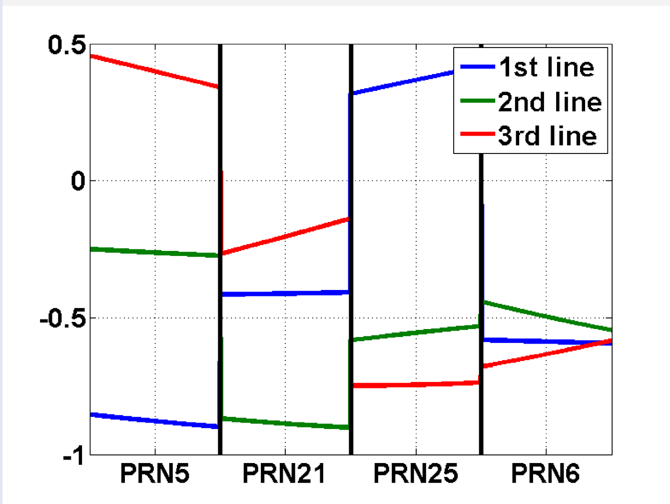

Fig.5 Lines of the design matrix

	1st estimate	2nd estimate	3rd estimate
W_equi: equivalent diagonal kernel based on the diagonal block of W_0	0,88%	0,75%	0,78%
W_diag: diagonal of W_0, in this case the identity matrix	18,60%	16,54%	20,20%

Tab.2 Relative error (%) for the 3 estimates for 2 different covariance matrices: W_equi and W_diag with respect to W_0

Conclusion

• A new diagonal equivalent kernel allows to take correlations into account in least-squares adjustment without using fully populated covariance matrices.
• For GPS design matrices, both the estimates and their cofactor matrices are more accurate (up to 20% for the estimates) than the commonly used diagonal elevation weighting models.

References

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