OPTIMAL REGULARIZATION FOR EXPONENTIALLY ILL-POSED PROBLEMS WITH STOCHASTICAL NOISE ESTIMATE

FRANK BAUER

Abstract. We consider the compact operator $A : \mathcal{X} \to \mathcal{Y}$ for the separable Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$. The problem $Ax = y$ is called exponentially ill-posed when the singular values $\sigma_k(A)$, $k = 1, 2, \ldots$ of the operator A tend to zero with exponential rate.

Classically one assumes that y is biased with "deterministic noise", i.e. that we have $y_\delta = y + \xi$ with $||\xi|| \leq \delta$. Instead we will assume to have "stochastical noise", i.e. $y_\delta = y + \xi$ with for all $f \in \mathcal{Y}$ we have that $\xi_f = \langle f, \xi \rangle_{\mathcal{Y}}$ is a Gaussian random variable fulfilling $\mathbb{E} \langle f, \xi \rangle_{\mathcal{Y}} = 0$ and $\mathbb{E} \langle f, \xi \rangle_{\mathcal{Y}}^2 = \delta^2 ||f||_{\mathcal{Y}}^2$.

This means that in terms of a Fourier expansion each Fourier coefficient is disturbed with Gaussian white noise. Regularization in this case is harder to perform than for the classical case.

For asymptotically optimal regularization with respect to $\delta \to 0$ we have two important parameters: the smoothness of x and the error behavior of $A^{-1}\xi$. It is well-known that on the one hand optimal regularization is possible when both smoothness and error behavior are known and impossible if none of them are known (Bakushinskii, 1984).

We will show that both for the "deterministic noise" case as well for the "stochastical noise" case we can regularize in an (asymptotically) optimal way just using the error behavior.

A practical application to the "downward-continuation" problem for satellite observed gravitational data in geosciences will be shown.

University of Kaiserslautern, Department of Mathematics, 67653 Kaiserslautern, P.O. Box 3049, Germany